

VERTEX-PANCYCLICITY OF HYPERTOURNAMENTS

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ABSTRACT. A hypertournament, or a k -tournament, on n vertices, $2 \leq k \leq n$, is a pair $T = (V, E)$, where the vertex set V is a set of size n and the edge set E is the collection of all possible subsets of size k of V , called the edges, each taken in one of its $k!$ possible permutations. A k -tournament is vertex-pancyclic if each vertex is contained in (directed) cycles of all possible lengths. A k -tournament is strong if there is a path from u to v for each pair of distinct vertices u and v . A question posed by Gutin and Yeo about the characterization of vertex-pancyclic hypertournaments is examined in this paper. We extend Moon's Theorem for tournaments to hypertournaments. We prove that if $k \geq 8$ and $n \geq k + 3$, then a k -tournament on n vertices is vertex-pancyclic if and only if it is strong. Similar results hold for other values of k . We also show that when $n \geq 7$, $k \geq 4$, and $n \geq k + 2$, a strong k -tournament on n vertices is pancyclic if and only if it is strong. The bound $n \geq k + 2$ is tight. We also find bounds for the generalized problem when we extend vertex-pancyclicity to require d edge-disjoint cycles of each possible length and extend strong connectivity to require d edge-disjoint paths between each pair of vertices. Our results include and extend that of Petrovic and Thomassen.

1. INTRODUCTION

Redei's Theorem and Camion's Theorem are two of the most well-known and important theorems regarding tournaments and Hamiltonicity.

Theorem 1 (Redei). *Every tournament has a Hamiltonian path.*

A tournament is d -edge-connected if, for any two distinct vertices $u, v \in V$, there are d pairwise edge-disjoint paths from u to v . It is *strongly connected*, or simply, *strong*, if it is 1-edge-connected.

Theorem 2 (Camion). *Every strong tournament has a Hamiltonian cycle.*

A tournament T is *vertex-pancyclic* if each vertex of T is contained in cycles of all possible lengths. Moon's Theorem generalizes Camion's Theorem.

Theorem 3 (Moon). *Every strong tournament is vertex-pancyclic.*

The proof of Redei's Theorem is very simple; and a proof of Moon's Theorem can be found in [1].

Let V be a n -set. Let E be the collection of all possible k -subsets of V , $2 \leq k \leq n$, each taken in one of its $k!$ possible permutations. A pair $T = (V, E)$ is called a *hypertournament*, or a *k -tournament*. Each element of V is a *vertex*, and each ordered k -tuple of E is a *hyperedge* or, simply, an *edge*.

For vertices $u, v \in V$ and an edge $e = (x_1, \dots, x_k) \in E$, we say u *dominates* v via edge e if u precedes v in e , that is if $u = x_i, v = x_j, 1 \leq i < j \leq k$. We denote this by uev . A *path* consists of an alternating sequence

$$x_0 e_1 x_1 e_2 x_2 \dots x_{\ell-1} e_\ell x_\ell$$

of distinct vertices x_i and distinct edges e_i so that x_{i-1} dominates x_i via $e_i, i = 1, \dots, \ell$. Such a path has *length* ℓ . A *cycle* is a path when all vertices are distinct except $x_0 = x_\ell$; the cycle has the same length ℓ . A path (cycle) of T is *Hamiltonian* if it contains all vertices of T . Let $V(X)$ and $E(X)$ denote the set of vertices and edges of X , respectively, where X could be a (hyper)tournament, path, or cycle.

It is natural for one to ask whether these theorems hold for hypertournaments as well. Gutin and Yeo [2] proved the following.

Theorem 4. *Let $k \geq 3$.*

- (i) *Every k -tournament on $n \geq k + 1$ vertices has a Hamiltonian path.*
- (ii) *Every strong k -tournament on $n \geq k + 2$ vertices has a Hamiltonian cycle.*

Recently, Petrovic and Thomassen [3] proved the following generalization of (ii).

Theorem 5. *Let T be a d -edge-connected k -tournament on n vertices. If $n \geq k + 1 + 24d$ for $k \geq 4$, and $n \geq 30d + 2$ for $k = 3$, then T has d edge-disjoint Hamiltonian cycles.*

Gutin and Yeo [2] mentioned as unsolved the problem of deciding if a k -tournament is pancyclic or vertex-pancyclic. Petrovic and Thomassen [3] characterized the vertex-pancyclic k -tournaments.

Theorem 6. *If $k \geq 4$ and $n \geq k + 25$ or if $k = 3$ and $n \geq 32$, then T is vertex-pancyclic if and only if T is strong.*

However, Petrovic and Thomassen’s characterization of vertex-pancyclic k -tournaments is incomplete in that it is only proved for sufficiently large n . In this paper, we will improve the bound that was given. Furthermore, regarding the pancyclicity question posed by Gutin and Yeo, we will give a necessary and sufficient condition for a k -tournament to be pancyclic. However, our characterization is only for $n \geq 7$ and $k \geq 4$. Finally, we will improve upon the generalization made by Petrovic and Thomassen in Theorem 5.

2. ON VERTEX-PANCYCLICITY

Petrovic and Thomassen answered the primary problem of deciding if a k -tournament is vertex-pancyclic for $n \geq k+25$, $k \geq 4$ and for $n \geq 32$, $k = 3$; we relax these restrictions of n . Their proof applied Hall’s Theorem first to provide disjointness of hyperedges, then they remove hyperedges used by the path P defined in Theorem 9 below. We reverse the order, applying Hall’s Theorem after removing the hyperedges of P .

Lemma 7. *Let T be a 3-tournament and P a path of T . A pair of distinct vertices x and y can be in at most four of the hyperedges of P .*

Proof. Let $P = v_0e_1v_1 \dots e_\ell v_\ell$. Suppose $x = v_m$ and $y = v_n$ (remember that vertices are distinct). As the hyperedges of a 3-tournament are triplets, if a hyperedge e_i of P contains both x and y , at least one of the vertices is an endpoint of the hyperedge in P . Therefore the hyperedges of P containing both x and y are all incident upon v_m or v_n in P . Hence there are at most four hyperedges of a path that contains a specific pair of distinct vertices.

If one of x and y is not a vertex of P (respectively, neither of them are), then it is clear that the maximum number of hyperedges containing both x and y is reduced to two (respectively, zero). □

Lemma 8. *If*

- (i) $k \geq 8$ and $n \geq k + 3$,
- (ii) $k \geq 5$ and $n \geq k + 4$,
- (iii) $k = 4$ and $n \geq 11$, or
- (iv) $k = 3$ and $n \geq 15$,

then

$$\binom{k}{2} \leq \left\lceil \frac{1}{2} \binom{n-2}{k-2} \right\rceil - (n-1) \quad \text{for } k \geq 4 \quad (*)$$

and

$$\binom{k}{2} \leq \left\lceil \frac{1}{2} \binom{n-2}{k-2} \right\rceil - 4 \quad \text{for } k = 3.$$

Proof. It is trivial to check the case $k = 3$. Assume $k \geq 4$. If $n = k + 3$, (*) can be rearranged to give

$$k^2 + k + 3 = 2 \left(\binom{k}{2} + k + 2 \right) - 1 \leq \binom{n-2}{k-2} = \binom{k+1}{k-2} = \frac{1}{6}(k^3 - k).$$

It is easy to verify that $k^3 - 6k^2 - 7k - 18 \geq 0$ for $k \geq 8$. Similarly, for $n = k + 4$, $k \geq 5$ suffices. For $k = 4$, (*) holds with $n = 11$. Now we apply induction on n ; we wish $\binom{n-2}{k-2} - 2\binom{k}{2} - 2n + 3 \geq 0$. We have established the respective base cases above. Increasing n by 1 increases $\binom{n-2}{k-2} - 2\binom{k}{2} - 2n + 3$ by $\binom{n-1}{k-2} - \binom{n-2}{k-2} - 2 = \binom{n-2}{k-3} - 2$. This is nonnegative for $k \geq 4$, $n \geq k + 1$. \square

Remark. The ceiling is actually required for the case of $k = 5$, $n = 9$; the inequality does not hold when the ceiling is removed.

We form an ordinary tournament (2-tournament) M from T with vertex set $V(T)$ in the following way. For $u, v \in V(T)$, orient the edge uv from u to v if u dominates v in at least half of the hyperedges of T containing u and v . M is called the *majority digraph* of T . In the case where u dominates v in exactly half of the hyperedges, M will have an edge from u to v and another edge from v to u . Therefore, it is possible that the majority digraph is not strictly a tournament. However, this distinction is immaterial to the following proofs. Alternatively, one can randomly choose one of the two edges to exclude from M to guarantee that the majority digraph is a tournament.

Theorem 9. *Let T be a k -tournament on n vertices. If*

- (i) $k \geq 8$ and $n \geq k + 3$,
- (ii) $k \geq 5$ and $n \geq k + 4$,
- (iii) $k = 4$ and $n \geq 11$, or
- (iv) $k = 3$ and $n \geq 15$,

then T is vertex-pancyclic if and only if T is strong.

Proof. It is obvious from the definition that a vertex-pancyclic T is strong. Now assume that T is strong. Fix a vertex x of T and a length $\ell \in \{3, 4, \dots, n\}$, we shall find an ℓ -cycle of T through x . By construction of M , if u dominates v in M , then u dominates v via $\left\lceil \frac{1}{2} \binom{n-2}{k-2} \right\rceil$ hyperedges of T . Call these the *corresponding hyperedges*.

Assume first that $k > 3$. If M is strong, then by Moon's Theorem M has an ℓ -cycle C' of M through x . Pick a corresponding hyperedge for each edge of C' , then we have an ℓ -cycle C in T . Thus we may assume that M is not strong.

The relation that two vertices u and v are strongly connected (that there exists a $u \rightarrow v$ path and a $v \rightarrow u$ path) is an equivalence relation; call the equivalence classes the *strong components*. If S_1 and S_2 are strong components, there cannot be both an $S_1 \rightarrow S_2$ edge and an $S_2 \rightarrow S_1$ edge. Hence we get a tournament on the strong components, treating each component as a vertex, and orienting the edges between vertices according to the direction of the edges between the strong components. By Redei's Theorem, there is a Hamiltonian path. Therefore, we can order the strong components of M as S_1, \dots, S_t such that there are no edges from S_j to S_i , $1 \leq i < j \leq t$. We say these strong components are *canonically ordered*. S_1 and S_t are called the *initial* and *terminal strong components* of M , respectively.

Because T is strong, there exists a path $P = x_0e_1x_1e_2x_2 \dots e_px_p$ in T connecting a vertex from the terminal component S_t to the initial component S_1 . Adding the p edges $\{x_0x_1, x_1x_2, \dots, x_{p-1}x_p\}$ to M , we obtain a strong semi-complete digraph D (every pair of vertices has one or two edges between them). As Moon's Theorem (a strong tournament is vertex-pancyclic) extends to strong semi-complete digraphs (see [1]), there exists an ℓ -cycle C' of D through x . We will form a cycle C of T from C' by using the same vertex set (in the same permutation). The only condition we need to check is that no edges are repeated. For an edge $x_{i-1}x_i$ of C' that originated from P , we use e_i for C ; note that these are distinct. For the remaining edges of C' , recall that each one has $\lceil \frac{1}{2} \binom{n-2}{k-2} \rceil$ corresponding hyperedges. We form a bipartite graph G with partite classes A and B . Every pair of vertices in T is a vertex in A . Every k -subset of vertices in T is a vertex in B . A vertex in A is joined to a vertex in B if the corresponding pair of vertices is contained in the corresponding k -subset. Since P has at most $n - 1$ hyperedges, after removing the hyperedges in P from B in the bipartite graph G , the vertices in A have degree at least $\lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - (n - 1)$. The vertices in B have degree $\binom{k}{2}$; thus by Hall's marriage Theorem, if $\binom{k}{2} \leq \lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - (n - 1)$, then there is a complete matching from A into B . By Lemma 8, the inequality is satisfied. Thus we can choose a hyperedge for each remaining edge of C' such that all the hyperedges chosen (including those from P) are distinct. Therefore by using these hyperedges, we have an ℓ -cycle C of T through x . As ℓ and x are arbitrary, T is vertex-pancyclic.

Assume now $k = 3$. We repeat the first part of the proof. Again, P may have as many as $n - 1$ hyperedges. However, by Lemma 7, at most 4 of the corresponding hyperedges for each pair of vertices are in

P . Therefore we require $\binom{k}{2} \leq \lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - 4$ instead, which is satisfied by Lemma 8. \square

Remark. Gutin and Yeo [2] proved that for $k \geq 3$, a hypertournament with $n \geq k + 1$ has a Hamiltonian path and one with $n \geq k + 2$ has a Hamiltonian cycle; see Theorem 4 above. These bounds are tight: It is obvious that there are no Hamiltonian paths for $n = k$; Gutin and Yeo constructed in [2] a strong k -tournament with $n = k + 1$ vertices yet admits no Hamiltonian cycle.

For $k \geq 8$, this theorem says a strong k -tournament with $n \geq k + 3$ vertices is vertex-pancyclic. We also know that this statement is false for $n = k + 1$. For the $n = k + 2$ case, it is currently unknown whether the statement holds. We will, however, fill this gap with pancyclicity in the following section. Also, for $k < 8$, there are a few more cases that have not been decided.

3. ON PANCYCLICITY

A k -tournament is *pancyclic* if it contains cycles of all possible lengths. For sufficiently large n and k , a strong k -tournament T with $n = k + 2$ vertices is pancyclic. To show this, we will modify Lemmas 3.2 and 3.4 of [2], which were used to prove the second part of Theorem 4.

Given a strong k -tournament T and its majority digraph M , we say that (P, Q) is an ℓ -cyclic pair of paths if P is a $x \rightarrow y$ path in T and Q is a $y \rightarrow x$ path in M such that $V(P) \cap V(Q) = \{x, y\}$ and $|V(P) \cup V(Q)| = \ell$. We also require that x and y be in different strong components of M if M is not strong.

For a path P and $u, v \in V(P)$, uPv is the segment of P from u to v . For paths P and Q such that the terminal vertex of P and the initial vertex of Q are the same or are separated by a single edge (as in the case of a 2-tournament), PQ is the path by adjoining Q after P .

Lemma 10. *For every strong k -tournament with n vertices, there exists an ℓ -cyclic pair of paths.*

Proof. Let T be a strong k -tournament with n vertices. Suppose the majority digraph M of T is strong. By Moon's Theorem, M is vertex-pancyclic; hence there exists an ℓ -cycle $R = x_1x_2 \dots x_\ell x_1$ in M . Then $P = x_1ex_2$, where e is a $x_1 \rightarrow x_2$ edge, and $Q = x_2Rx_1$ form an ℓ -cyclic pair of paths.

Now we assume that M is not strong, and let S_1, \dots, S_t be the canonically ordered strong components of M (as in Theorem 9). Let $R = x_1e_1x_2e_2 \dots e_{m-1}x_m$ be the shortest $S_t \rightarrow S_1$ path in T . Then $x_1 \in S_t$, $x_m \in S_1$, and $\{x_2, \dots, x_{m-1}\} \cap (S_1 \cup S_t) = \emptyset$. If $m \geq \ell$, then

x_ℓ is not in S_t , and that there are no edges from S_t , which contains x_1 , to the strong component containing x_ℓ . Therefore, since M is semi-complete, $x_\ell x_1$ is an edge in M . Then $P = x_1 R x_\ell$ and $Q = x_\ell x_1$ form an ℓ -cyclic pair of paths, where the endpoints are in different strong components.

Now assume $m < \ell$, let $M' = M - \{x_2, x_3, \dots, x_{m-1}\}$ and let $S'_1, \dots, S'_{t'}$ be the canonically ordered strong components of M' . Notice that M' is also semi-complete, and that $x_1 \in S'_{t'}$ and $x_m \in S'_1$ are still in the (new) terminal and initial components, respectively. We want a path Q of length $\ell - m + 1 \geq 2$ from x_m to x_1 in M' ; then (R, Q) is an ℓ -cyclic pair of paths.

It remains to construct the desired path Q . Let j be minimum such that $s = |S'_1| + |S'_2| + \dots + |S'_j| + |S'_{j+1}| \geq \ell - m + 1$. For each strong component S'_i , $1 \leq i \leq j$, we choose a Hamiltonian path Q'_i , whose existence is guaranteed by Camion's Theorem. For Q'_1 , we stipulate that the path starts at $x_m \in S'_1$. Now we construct a path Q'_{j+1} such that $Q = x_m Q'_1 Q'_2 \dots Q'_j Q'_{j+1} x_1$ is of length $\ell - m + 1$. Notice that there is precisely one edge from the terminal vertex of Q'_i to the initial vertex of Q'_{i+1} , so the path Q is well-defined. If $j + 1 = t'$, then $x_1 \in S'_{j+1}$ and S'_{j+1} is strong, hence vertex-pancyclic (Moon's Theorem). Thus we can construct a path Q'_{j+1} in S'_{j+1} with $\ell - m + 2 - s + |S'_{j+1}|$ vertices, ending at x_1 . Otherwise, if $j + 1 < t'$, let Q'_{j+1} be a path in S'_{j+1} with $\ell - m + 1 - s + |S'_{j+1}|$ vertices. Then indeed $|Q'_1| + |Q'_2| + \dots + |Q'_j| + |Q'_{j+1} \cup \{x_1\}| = \ell - m + 2$, so Q has length $\ell - m + 1$, as desired. \square

For distinct vertices $u, v \in V(T)$, let $E_T(u, v)$ denote the set of edges of $E(T)$ in which u dominates v ; the subscript T is omitted when the tournament is clear from context.

Theorem 11. *Every strong k -tournament with n vertices, where $n \geq k + 2 \geq 6$ and $n \geq 7$, is pancyclic.*

Proof. For $n \geq k + 2 \geq 6$, we have $\binom{n-2}{k-2} \geq 2n - 4$ if and only if $n \geq 7$. Let T be a k -tournament with n vertices such that $n \geq k + 2 \geq 6$ and $\binom{n-2}{k-2} \geq 2n - 4$, and let M be the majority digraph of T .

Gutin and Yeo proved that T is Hamiltonian (Theorem 4). Thus we fix a length $\ell \in \{3, 4, \dots, n - 1\}$, and show that there exists an ℓ -cycle in T .

We first suppose that M is strong. By Moon's Theorem, there exists an ℓ -cycle $C' = x_1 x_2 \dots x_\ell x_1$ in M . For $i = 1, 2, \dots, \ell$, we have $|E(x_{i-1}, x_i)| \geq \frac{1}{2} \binom{n-2}{k-2} \geq n - 2$, where we define $x_0 = x_\ell$. If $\ell \leq n - 2$, we can choose corresponding hyperedges e_j from T so

that $C = x_1e_1x_2e_2\dots e_{\ell-1}x_\ell e_\ell x_1$ is an ℓ -cycle in T . So we may assume that $\ell = n - 1$. There exist distinct edges e_1 and e_2 , such that $\{e_1, e_2\} \subseteq E(x_1, x_2)$. Also, since $k \leq n - 2 = \ell - 1$, and since e_1 and e_2 does not contain the same set of vertices, one of e_1, e_2 does not contain a vertex in the set $\{x_3, x_4, \dots, x_{\ell-1}\}$. Without loss of generality, assume $x_i \notin e_1$, where $i \in \{3, 4, \dots, \ell - 1\}$. Since $|E(x_{i-1}, x_i)| \geq \ell - 1$, we can choose corresponding hyperedges f_j from T so that

$$P = x_i f_1 x_{i+1} f_2 \dots x_\ell f_{\ell-i+1} x_1 e_1 x_2 f_{\ell-i+2} x_3 \dots f_{\ell-2} x_{i-1}$$

is a path of length $\ell - 1$. Since $x_i \notin e_1$, we have $e_1 \notin E(x_{i-1}, x_i)$; since $|E(x_{i-1}, x_i)| \geq \ell - 1$, there is an edge $f_{\ell-1} \in E(x_{i-1}, x_i) - E(P)$. Hence $C = P x_{i-1} f_{\ell-1} x_i$ is a cycle of length ℓ , as desired.

Now suppose that M is not strong. By Lemma 10, there exists an ℓ -cyclic pair of paths (P, Q) , where $P = x_1 e_1 x_2 e_2 \dots e_{p-1} x_p$ is a path in T , and $Q = y_1 y_2 \dots y_q$ is a path in M . Recall that $y_1 = x_p$, $y_q = x_1$, and y_1, y_q are in different strong components of M . Fix i such that y_i and y_{i+1} are in different strong components of M . By definition of M , we have $|E(y_{j-1}, y_j)| \geq \frac{1}{2} \binom{n-2}{k-2} \geq n - 2 \geq \ell - 1$ for $j > 1$. Also, if $|E(y_i, y_{i+1})| = n - 2$, then $|E(y_{i+1}, y_i)| \geq n - 2$; but y_i and y_{i+1} are in different strong components, thus $|E(y_i, y_{i+1})| \geq n - 1 \geq \ell$. As $|E(y_{j-1}, y_j)| \geq \ell - 1$ for $j > 1$, we can extend the path P to a path $R = r_1 f_1 r_2 f_2 \dots f_{\ell-1} r_\ell$ in T with $r_1 = y_{i+1}$, $r_2 = y_{i+2}, \dots, r_{q-i} = y_q = x_1$, $r_{q-i+1} = x_2, \dots, r_{\ell+1-i} = x_p = y_1$, $r_{\ell+2-i} = y_2, \dots, r_\ell = y_i$, using edges of P and choosing the remaining edges. Now as $|E(y_i, y_{i+1})| \geq \ell$, there is an edge $f_\ell \in E(y_i, y_{i+1}) - E(R)$. Hence $R y_i f_\ell y_{i+1}$ is a cycle of length ℓ in T . \square

4. ON d -DISJOINT-VERTEX-PANCYCLICITY

A k -tournament is d -disjoint-vertex-pancyclic if each vertex of T is contained in d edge-disjoint ℓ -cycles for each possible length ℓ . Now we consider the generalized problem by extending vertex-pancyclicity to require d edge-disjoint cycles of each possible length and extending strong connectivity to require d edge-disjoint paths between each pair of vertices.

To prove the following lemma, we use the calculus of finite differences: The *finite forward difference* $\Delta_x f(x)$ of a function $f(x)$ with respect to x is defined as $\Delta_x f(x) = f(x + 1) - f(x)$. The n -th finite difference is defined inductively as $\Delta_x^n f(x) = \Delta_x(\Delta_x^{n-1} f(x))$. If the function has more variables, they are held constant. This is the discrete analog of the derivative.

Lemma 12. *If $n \geq k + 2d + 1$ for $k \geq 8$, then*

$$d \left[\binom{k}{2} + (n - 1) \right] \leq \left\lceil \frac{1}{2} \binom{n-2}{k-2} \right\rceil.$$

Proof. Suppose $k \geq 8$ and $n \geq k + 2d + 1$, we want to show that

$$f = f(n, k, d) = \binom{n-2}{k-2} + 1 - 2d \left[\binom{k}{2} + n - 1 \right]$$

is nonnegative. Now

$$\begin{aligned} \Delta_n f(n, k, d) &= \binom{n-1}{k-2} + 1 - 2d \left[\binom{k}{2} + n \right] \\ &\quad - \left(\binom{n-2}{k-2} + 1 - 2d \left[\binom{k}{2} + n - 1 \right] \right) \\ &= \binom{n-2}{k-2} + \binom{n-2}{k-3} - 2dn - \left(\binom{n-2}{k-2} - 2dn + 2d \right) \\ &= \binom{n-2}{k-3} - 2d, \end{aligned}$$

which is nonnegative when $n \geq k + 2d + 1$ and $k \geq 8$. Therefore it remains to show that f is nonnegative when $n = k + 2d + 1$.

To show that

$$g(k, d) = f(k + 2d + 1, k, d) = \binom{k+2d-1}{k-2} + 1 - 2d \left[\binom{k}{2} + k + 2d \right]$$

is nonnegative, we shall show that $g(k, 1)$, $\Delta_d g(k, 1)$, $\Delta_d^2 g(k, 1)$ and $\Delta_d^3 g(k, d)$ are all nonnegative for $k \geq 8$.

Now $g(k, 1) = \binom{k+1}{k-2} + 1 - 2 \left[\binom{k}{2} + k + 3 \right]$ reduces to the case of $n = k + 3$ considered in Lemma 8, and is nonnegative.

Now note that

$$\binom{k+2d+1}{k-2} = \binom{k+2d+1}{2d+3} = \frac{k+2d+1}{2d+3} \frac{k+2d}{2d+2} \binom{k+2d-1}{2d+1},$$

hence

$$\begin{aligned} \Delta_d g(k, d) &= g(k, d+1) - g(k, d) \\ &= \binom{k+2d+1}{k-2} - 2 \left[\binom{k}{2} + k \right] - 4(d+1)^2 - \binom{k+2d-1}{2d+1} + 4d^2 \\ &= \left(\frac{k+2d+1}{2d+3} \frac{k+2d}{2d+2} - 1 \right) \binom{k+2d-1}{2d+1} - k^2 - k - 8d - 4. \end{aligned}$$

In particular,

$$\begin{aligned} \Delta_d g(k, 1) &= \left(\frac{k+3}{5} \frac{k+2}{4} - 1 \right) \binom{k+1}{3} - k^2 - k - 8 - 4 \\ &= \frac{1}{120} (k^5 + 5k^4 - 15k^3 - 125k^2 - 106k - 1440), \end{aligned}$$

which has only one real root between 5 and 6, and is 37 when $k = 6$. Similarly,

$$\begin{aligned} \Delta_d^2 g(k, d) &= \Delta_d g(k, d+1) - \Delta_d g(k, d) \\ &= \left[\left(\frac{k+2d+3}{2d+5} \frac{k+2d+2}{2d+4} - 2 \right) \left(\frac{k+2d+1}{2d+3} \frac{k+2d}{2d+2} \right) + 1 \right] \binom{k+2d-1}{2d+1} - 8. \end{aligned}$$

Then

$$\begin{aligned}\Delta_d^2 g(k, 1) &= \left[\left(\frac{k+5}{7} \frac{k+4}{6} - 2 \right) \left(\frac{k+3}{5} \frac{k+2}{4} \right) + 1 \right] \binom{k+1}{3} - 8 \\ &= \frac{1}{5040} (k^2 + 19k + 76)(k+1)k(k-1)(k-2)(k-3) - 8,\end{aligned}$$

which is clearly increasing when $k \geq 4$, and is 20 when $k = 5$. Finally,

$$\begin{aligned}\Delta_d^3 g(k, d) &= \Delta_d^2 g(k, d+1) - \Delta_d^2 g(k, d) \\ &= \frac{(k^2+8dk+17k+16d^2+56d+30)(k+4d+7)(k-2)(k-3)(k-4)}{(2d+7)(2d+6)(2d+5)(2d+4)(2d+3)(2d+2)} \binom{k+2d-1}{2d+1},\end{aligned}$$

which is clearly nonnegative for $k \geq 4$, $d \geq 1$. Therefore $g(k, d) \geq 0$ for $k \geq 8$, and $f(n, k, d)$ is increasing in n for $n \geq k + 2d + 1$, we have that f is indeed nonnegative for $n \geq k + 2d + 1$, $k \geq 8$, as desired. \square

Theorem 13. *Let T be a k -tournament on n vertices. If*

- (i) $k \geq 8$ and $n \geq k + 2d + 1$, or
- (ii) $k = 3$ and $n \geq 14d + 1$,

then T is d -disjoint-vertex-pancyclic if and only if T is d -edge-connected.

Proof. We review the proof of Theorem 9. It is again obvious that a d -disjoint-vertex-pancyclic T is d -edge-connected. Now assume that T is d -edge-connected. We again consider $k > 3$ first. Fix a vertex x of T and a length $\ell \in \{3, \dots, n\}$, we shall find d disjoint ℓ -cycles of T through x . We again construct the majority digraph M . Previously, since T is strong, there exists a path P in T connecting a vertex x_1 from the terminal component S_t to a vertex x_2 in the initial component S_1 . Now, because T is d -edge-connected, T has d edge-disjoint paths P_1, P_2, \dots, P_d connecting x_1 to x_2 . We add the edges of P_i to M to obtain a strong semi-complete digraph D_i . Moon's Theorem gives us an ℓ -cycle C'_i through x . Now we choose corresponding hyperedges to get d disjoint ℓ -cycles of T through x . The edges arising only from the P_i can be directly used, as before. Since P_i has at most $n - 1$ hyperedges, P_1, \dots, P_d can use at most $d(n - 1)$ hyperedges. Since each remaining edge of M corresponds to $\lceil \frac{1}{2} \binom{n-2}{k-2} \rceil$ hyperedges, after removing the hyperedges in the P_i from B in the bipartite graph G , the vertices in A have degree at least $\lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - d(n - 1)$. Now we want to choose d corresponding hyperedges for each of the remaining edges of M such that all edges are disjoint. This can be accomplished by replicating A a total of d times. Let $A' = \{(a, i) : a \in A, 1 \leq i \leq d\}$, and join $(a, i) \in A'$ with $b \in B$ if and only if a and b are joined in G . Consider this new bipartite graph with partite classes A' and B . Now each vertex in B has degree $d \binom{k}{2}$. Since

$$d \binom{k}{2} \leq \lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - d(n - 1)$$

by Lemma 12, Hall's Marriage Theorem ensures that there exists a complete matching from A' into B . Thus for each edge e of M , we get d hyperedges of T by taking those matched to (e, i) , $1 \leq i \leq d$. Now as these hyperedges are all distinct, and disjoint from those of P_1, \dots, P_d , we can use these to form edge-disjoint ℓ -cycles C_1, \dots, C_d through x in T .

For $k = 3$, Lemma 7 allows us to replace $n - 1$ with 4 in the above proof. Now $d \binom{k}{2} \leq \lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - 4d$ becomes $3d \leq \lceil \frac{1}{2}(n - 2) \rceil - 4d$, which is satisfied when $n \geq 14d + 1$. \square

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