HAMILTONIAN CYCLES IN TOURNAMENTS

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1. Definitions

Let V be a n-set (set of size n). Let E be the collection of all possible k-subsets (subsets of size k) of V, $2 \le k \le n$, each taken in one of its $k!$ possible permutations. A pair $T = (V, E)$ is called a *hypertournament*, or a *k-tournament*. Each element of V is a *vertex*, and each ordered k-tuple of E is a *hyperedge* or, simply, an *edge*. For vertices $u, v \in V$ and an edge $e = (x_1, \dots, x_k) \in E$, we say u *dominates* v via edge e if u precedes v in e, that is if $u = x_i$, $v = x_j$, $1 \le i < j \le k$. We denote this by uev. A *path* consists of an alternating sequence

$$x_0 e_1 x_1 e_2 x_2 \dots x_{\ell-1} e_\ell x_\ell$$

of distinct vertices x_i and distinct edges e_i so that x_{i-1} dominates x_i via e_i, $i = 1, \dots, \ell$. Such a path has *length* ℓ. A *cycle* is a path when all vertices are distinct except $x_0 = x_\ell$. A path (cycle) of T is *Hamiltonian* if it contains all vertices of T. A k-tournament T is *pancyclic* if it contains cycles of all possible lengths. It is *vertex-pancyclic* if each vertex of T is contained in cycles of all possible lengths. A k-tournament T is *strong* if there is a path from u to v for each pair $u, v \in V$. The vertex set V and the edge collection E is also denoted $V(T)$ and $E(T)$, respectively. A hypertournament T is *d-edge-connected* if, for any two vertices u, $v \in V$, there are d pairwise edge-disjoint paths from x to y. For an ordinary graph (V, E), $\Gamma(v)$ is the neighbors of v. For $A \subseteq V$, we denote by $\Gamma(A)$ the set $\bigcup_{a \in A} \Gamma(a)$.

2. Background and Motivation

Gutin and Yeo [2] proved the following.

Theorem 1. (i) *Every k-tournament on n vertices, $3 \le k \le n-1$, has a Hamiltonian path.*
(ii) *Every strong k-tournament on n vertices, $3 \le k \le n-2$, has a Hamiltonian cycle.*

Recently, Petrovic and Thomassen [4] proved the following generalization of (ii).

Theorem 2. *Let T be a d-edge-connected k-tournament on n vertices. If $n \geq k + 1 + 24d$ for $k \geq 4$, and $n \geq 30d + 2$ for $k = 3$, then T has d edge-disjoint Hamiltonian cycles.*

Gutin and Yeo [2] mentioned as unsolved the problem of deciding if a k-tournament is pancyclic. Petrovic and Thomassen [4] characterized the pancyclic k-tournaments: If $k \geq 4$ and $n \geq k + 25$ or if $k = 3$ and $n \geq 32$, then T is vertex-pancyclic if and only if T is strong.

This characterization is incomplete in that it is only for n large compared to k. Furthermore, this is claimed without explaination. Therefore, I shall deduce this claim from the proof of Theorem 2, in the case when $d = 1$. Then I shall attempt to solve the general problem for n not necessarily sufficiently large. The work of Petrovic and Thomassen [4] is very recent (2006), and therefore fits into ongoing work in mathematics.

3. Progress

The primary problem we are working on is to characterize vertex-pancyclic k-tournaments on n vertices. I read the primary reference [4] of my problem and read [2], where the original problem was posed. I understood the proof of Moon's theorem (every strong tournament is vertex pancyclic) from [1], and deduced Camion's theorem (every strong tournament has a Hamiltonian cycle). I independently proved Redei's theorem (every tournament has a Hamiltonian path) and presented the proof to my mentor Professor Wilson. I found additional sources [3, 6, 5]. I used the insight gained from [2] to understand the proof of Theorem 2 from [4]. With a few modifications, Theorem 2 implies the result of Petrovic and Thomassen [4] regarding the primary problem.

For the case of $k = 3$, Petrovic and Thomassen answered the primary problem for $n \geq 32$, we answer it for $n \geq 29$.

Lemma 3. *Let T be a 3-tournament and P a path of T. A pair of distinct vertices x and y can be in at most four of the hyperedges of P.*

Proof. Let $P = v_0 e_1 v_1 \ldots e_\ell v_\ell$. Suppose $x = v_m$ and $y = v_n$ (remember that vertices are distinct). As the hyperedges of a 3-tournament are triplets, if a hyperedge e_i of P contains both x and y, at least one of the vertices is an endpoint of the hyperedge in P. Therefore the hyperedges of P containing both x and y are all incident upon v_m or v_n

in P. Hence there are at most four hyperedges of a path that contains a specific pair of distinct vertices.

If one of x and y is not a vertex of P (respectively neither of them are), then it is clear that the maximum number of hyperedges containing both x and y is reduced to two (respectively zero). □

Now we slightly alter Lemma 1 from [4]. Given a 3-tournament T, we form a bipartite graph G with partite classes A and B. Every pair of vertices in T is a vertex in A. Every 3-subset of vertices in T is a vertex in B. A vertex in A is joined to a vertex in B if the corresponding pair of vertices is contained in the corresponding k-subset.

Lemma 4. *If $n \geq 29$, then the bipartite graph G contains a spanning subgraph G' such that every vertex in A has degree 9 and every vertex in B has degree at most 1 in G'.*

Proof. Every vertex in A has degree $n-2 \geq 27$ in G, and every vertex in B has degree 3 in G. It follows that, for any subset $S \subseteq A$, $|\Gamma(S)| \geq \frac{n-2}{3}|S| \geq 9|S|$. Hence G' exists by Hall's marriage theorem. □

Using G' we form an ordinary tournament (2-tournament) T' from T with vertex set $V(T)$ in the following way. For $u, v \in V(T)$, orient the edge uv from u to v iff u dominates v in at least 5 out of 9 neighbors of $\{u, v\}$ in G'.

Theorem 5. *Let T be a 3-tournament on n vertices. If $n \geq 29$, then T is vertex-pancyclic if and only if T is strong.*

Proof. It is obvious from the definition that a vertex-pancyclic T is strong. Now assume that T is strong. Fix a vertex x of T and a length $\ell \in \{3, \ldots, n\}$, we shall find an ℓ-cycle of T through x. By construction of T', if u dominates v in T', then u dominates v via 5 hyperedges of T. Call these the "corresponding hyperedges." Moreover, these $5\binom{n}{2}$ hyperedges are distinct by Lemma 4.

If T' is strong, then by Camion's theorem T' has a Hamiltonian cycle H'. Pick a corresponding hyperedge for each edge of H', then we have a Hamiltonian cycle H in T. Thus we may assume that T' is not strong.

The relation that two vertices u and v are strongly connected (that there exists a $u \to v$ path and a $v \to u$ path) is an equivalence relation; call the equivalence classes the strong components. If S_1 and S_2 are strong components, there cannot be both an $S_1 \to S_2$ edge and an $S_2 \to S_1$ edge. Hence we get a tournament on the strong components, treating each component as a vertex, and orienting the edges between vertices according to the direction of the edges between the strong components. By Redei's theorem, there is a Hamiltonian path.

Therefore, we can order the strong components of T' as $T'_1, \ldots, T'_t$ such that there are no hyperedges from T'_j to T'_i, $1 \leq i < j \leq t$.

Because T is strong, there exists a path $P = x_0e_1x_1e_2x_2 \ldots e_px_p$ in T connecting a vertex from the terminal component T'_t to the initial component T'_1. Adding the p edges $\{x_0x_1, x_1x_2, \ldots, x_{p-1}x_p\}$ to T', we obtain a strong semi-complete digraph D (every pair of vertices has one or two edges between them). As Moon's theorem (a strong tournament is vertex-pancyclic) extends to strong semi-complete digraphs (see [1]), there exists an ℓ-cycle C' of D through x. We will form a cycle C of T from C' by using the same vertex set (in the same permutation). The only condition we need to check is that no edges are repeated. For an edge $x_{i-1}x_i$ of C' that originated from P, we use e_i for C; note that these are distinct. For the remaining edges of C', recall that each one has 5 corresponding hyperedges. By Lemma 3, at most 4 of these 5 hyperedges are in P. Therefore, for each of the remaining edges, there exists a corresponding hyperedge that is disjoint from those of P. These edges are themselves distinct by construction of G' (vertices in B has degree at most 1; Lemma 4). Therefore by using these hyperedges, we have an ℓ-cycle C of T through x. As ℓ and x are arbitrary, T is vertex-pancyclic. □

4. Plans and Remarks

In the coming month, we plan to continue improving the partial result to the primary problem. There are several steps of the proof of Theorem 2 that may be improved, especially for the special case of $d = 1$.

In reading [4], the lack of clear definitions of terminology made the proofs difficult to follow. We utilized [2] as a resource to understand the terminology. In the future, if similar situations arise, we shall read references provided by the paper in question to further understand both terminology used and techniques employed. Thus the resources needed will be the vast journal archives that the Caltech library system provides. Another challenge is to know the proper direction of approach to unsolved problems.

References

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