

# VERTEX-PANCYCLICITY OF HYPERTOURNAMENTS

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## 1. DEFINITIONS

Let  $V$  be a  $n$ -set (set of size  $n$ ). Let  $E$  be the collection of all possible  $k$ -subsets (subsets of size  $k$ ) of  $V$ ,  $2 \leq k \leq n$ , each taken in one of its  $k!$  possible permutations. A pair  $T = (V, E)$  is called a *hypertournament*, or a *k-tournament*. Each element of  $V$  is a *vertex*, and each ordered  $k$ -tuple of  $E$  is a *hyperedge* or, simply, an *edge*. For vertices  $u, v \in V$  and an edge  $e = (x_1, \dots, x_k) \in E$ , we say  $u$  *dominates*  $v$  via edge  $e$  if  $u$  precedes  $v$  in  $e$ , that is if  $u = x_i, v = x_j, 1 \leq i < j \leq k$ . We denote this by  $uev$ . A *path* consists of an alternating sequence

$$x_0 e_1 x_1 e_2 x_2 \dots x_{\ell-1} e_{\ell} x_{\ell}$$

of distinct vertices  $x_i$  and distinct edges  $e_i$  so that  $x_{i-1}$  dominates  $x_i$  via  $e_i, i = 1, \dots, \ell$ . Such a path has *length*  $\ell$ . A *cycle* is a path when all vertices are distinct except  $x_0 = x_{\ell}$ . A path (cycle) of  $T$  is *Hamiltonian* if it contains all vertices of  $T$ . A  $k$ -tournament  $T$  is *pancyclic* if it contains cycles of all possible lengths. It is *vertex-pancyclic* if each vertex of  $T$  is contained in cycles of all possible lengths. It is *d-disjoint-vertex-pancyclic* if each vertex of  $T$  is contained in  $d$  edge-disjoint  $\ell$ -cycles for each possible length  $\ell$ . A  $k$ -tournament  $T$  is *strong* if there is a path from  $u$  to  $v$  for each pair  $u, v \in V$ . The vertex set  $V$  and the edge collection  $E$  is also denoted  $V(T)$  and  $E(T)$ , respectively. A hypertournament  $T$  is *d-edge-connected* if, for any two vertices  $u, v \in V$ , there are  $d$  pairwise edge-disjoint paths from  $u$  to  $v$ . For an ordinary graph  $(V, E)$ ,  $\Gamma(v)$  is the neighbors of  $v$ . For  $A \subseteq V$ , we denote by  $\Gamma(A)$  the set  $\bigcup_{a \in A} \Gamma(a)$ .

## 2. KNOWN RESULTS

**Theorem 1** (Redei). *Every tournament has a Hamiltonian path.*

**Theorem 2** (Moon). *Every strong tournament is vertex-pancyclic.*

Gutin and Yeo [2] proved the following.

**Theorem 3.** *Let  $k \geq 3$ .*

- (i) *Every  $k$ -tournament on  $n \geq k + 1$  vertices has a Hamiltonian path.*
- (ii) *Every strong  $k$ -tournament on  $n \geq k + 2$  vertices has a Hamiltonian cycle.*

Recently, Petrovic and Thomassen [4] proved the following generalization of (ii).

**Theorem 4.** *Let  $T$  be a  $d$ -edge-connected  $k$ -tournament on  $n$  vertices. If  $n \geq k + 1 + 24d$  for  $k \geq 4$ , and  $n \geq 30d + 2$  for  $k = 3$ , then  $T$  has  $d$  edge-disjoint Hamiltonian cycles.*

Gutin and Yeo [2] mentioned as unsolved the problem of deciding if a  $k$ -tournament is pancyclic. Petrovic and Thomassen [4] characterized the pancyclic  $k$ -tournaments: If  $k \geq 4$  and  $n \geq k + 25$  or if  $k = 3$  and  $n \geq 32$ , then  $T$  is vertex-pancyclic if and only if  $T$  is strong.

### 3. PROGRESS

Petrovic and Thomassen answered the primary problem for  $n \geq k + 25$ ,  $k \geq 4$  and  $n \geq 32$ ,  $k = 3$ , we relax these restrictions of  $n$ . Their proof applied Hall's theorem first to provide disjointness, then remove hyperedges used by the path  $P$ . We reverse the order, applying Hall's theorem after removing the hyperedges of  $P$ .

**Lemma 5.** *Let  $T$  be a 3-tournament and  $P$  a path of  $T$ . A pair of distinct vertices  $x$  and  $y$  can be in at most four of the hyperedges of  $P$ .*

*Proof.* Let  $P = v_0 e_1 v_1 \dots e_\ell v_\ell$ . Suppose  $x = v_m$  and  $y = v_n$  (remember that vertices are distinct). As the hyperedges of a 3-tournament are triplets, if a hyperedge  $e_i$  of  $P$  contains both  $x$  and  $y$ , at least one of the vertices is an endpoint of the hyperedge in  $P$ . Therefore the hyperedges of  $P$  containing both  $x$  and  $y$  are all incident upon  $v_m$  or  $v_n$  in  $P$ . Hence there are at most four hyperedges of a path that contains a specific pair of distinct vertices.

If one of  $x$  and  $y$  is not a vertex of  $P$  (respectively neither of them are), then it is clear that the maximum number of hyperedges containing both  $x$  and  $y$  is reduced to two (respectively zero).  $\square$

**Lemma 6.** *If  $n \geq k + 3$  for  $k \geq 8$ ,  $n \geq k + 4$  for  $k \geq 5$ ,  $n \geq 11$  for  $k = 4$ , and  $n \geq 15$  for  $k = 3$ , then  $\binom{k}{2} \leq \lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - (n-1)$  for  $k \geq 4$  and  $\binom{k}{2} \leq \lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - 4$  for  $k = 3$ .*

*Proof.* It is trivial to check the case of  $k = 3$ . Assume  $k \geq 4$ . Suppose  $n = k + 3$ , the inequality can be rearranged to give  $k^2 + k + 3 =$

$2\binom{k}{2} + k + 2 - 1 \leq \binom{n-2}{k-2} = \binom{k+1}{k-2} = \frac{1}{6}(k^3 - k)$ . It is easy to verify that  $k^3 - 6k^2 - 7k - 18 \geq 0$  for  $k \geq 8$ . Similarly, for  $n = k + 4$ ,  $k \geq 5$  suffices. For  $k = 4$ , the inequality holds with  $n = 11$ . Now we apply induction on  $n$ , we wish  $\binom{n-2}{k-2} - 2\binom{k}{2} - 2n + 3 \geq 0$ . We have established the respective base cases above. If this inequality holds, increasing  $n$  by 1 increases  $\binom{n-2}{k-2} - 2\binom{k}{2} - 2n + 3$  by  $\binom{n-1}{k-2} - \binom{n-2}{k-2} - 2 = \binom{n-2}{k-3} - 2$ . This is nonnegative for  $k \geq 4$ ,  $n \geq k + 1$ .  $\square$

*Remark.* The ceiling is actually required for the case of  $k = 5$ ,  $n = 9$ ; the inequality does not hold when the ceiling is removed.

Given a  $k$ -tournament  $T$ , we form a bipartite graph  $G$  with partite classes  $A$  and  $B$ . Every pair of vertices in  $T$  is a vertex in  $A$ . Every  $k$ -subset of vertices in  $T$  is a vertex in  $B$ . A vertex in  $A$  is joined to a vertex in  $B$  if the corresponding pair of vertices is contained in the corresponding  $k$ -subset.

We form an ordinary tournament (2-tournament)  $T'$  from  $T$  with vertex set  $V(T)$  in the following way. For  $u, v \in V(T)$ , orient the edge  $uv$  from  $u$  to  $v$  if  $u$  dominates  $v$  in at least half of the hyperedges of  $T$  containing  $u$  and  $v$ .  $T'$  is called the majority digraph of  $T$ .

**Theorem 7.** *Let  $T$  be a  $k$ -tournament on  $n$  vertices. If  $n \geq k + 3$  for  $k \geq 8$ ,  $n \geq k + 4$  for  $k \geq 5$ ,  $n \geq 11$  for  $k = 4$ , and  $n \geq 15$  for  $k = 3$ , then  $T$  is vertex-pancyclic if and only if  $T$  is strong.*

*Proof.* It is obvious from the definition that a vertex-pancyclic  $T$  is strong. Now assume that  $T$  is strong. Fix a vertex  $x$  of  $T$  and a length  $\ell \in \{3, \dots, n\}$ , we shall find an  $\ell$ -cycle of  $T$  through  $x$ . By construction of  $T'$ , if  $u$  dominates  $v$  in  $T'$ , then  $u$  dominates  $v$  via  $\lceil \frac{1}{2} \binom{n-2}{k-2} \rceil$  hyperedges of  $T$ . Call these the ‘‘corresponding hyperedges.’’

Assume first that  $k > 3$ . If  $T'$  is strong, then by Moon’s theorem  $T'$  has an  $\ell$ -cycle  $C'$  of  $T'$  through  $x$ . Pick a corresponding hyperedge for each edge of  $C'$ , then we have an  $\ell$ -cycle  $C$  in  $T$ . Thus we may assume that  $T'$  is not strong.

The relation that two vertices  $u$  and  $v$  are strongly connected (that there exists a  $u \rightarrow v$  path and a  $v \rightarrow u$  path) is an equivalence relation; call the equivalence classes the strong components. If  $S_1$  and  $S_2$  are strong components, there cannot be both an  $S_1 \rightarrow S_2$  edge and an  $S_2 \rightarrow S_1$  edge. Hence we get a tournament on the strong components, treating each component as a vertex, and orienting the edges between vertices according to the direction of the edges between the strong components. By Redei’s theorem, there is a Hamiltonian path. Therefore, we can order the strong components of  $T'$  as  $T'_1, \dots, T'_t$  such that there are no hyperedges from  $T'_j$  to  $T'_i$ ,  $1 \leq i < j \leq t$ .

Because  $T$  is strong, there exists a path  $P = x_0e_1x_1e_2x_2 \dots e_px_p$  in  $T$  connecting a vertex from the terminal component  $T'_t$  to the initial component  $T'_1$ . Adding the  $p$  edges  $\{x_0x_1, x_1x_2, \dots, x_{p-1}x_p\}$  to  $T'$ , we obtain a strong semi-complete digraph  $D$  (every pair of vertices has one or two edges between them). As Moon's theorem (a strong tournament is vertex-pancyclic) extends to strong semi-complete digraphs (see [1]), there exists an  $\ell$ -cycle  $C'$  of  $D$  through  $x$ . We will form a cycle  $C$  of  $T$  from  $C'$  by using the same vertex set (in the same permutation). The only condition we need to check is that no edges are repeated. For an edge  $x_{i-1}x_i$  of  $C'$  that originated from  $P$ , we use  $e_i$  for  $C$ ; note that these are distinct. For the remaining edges of  $C'$ , recall that each one has  $\lceil \frac{1}{2} \binom{n-2}{k-2} \rceil$  corresponding hyperedges. Since  $P$  has at most  $n - 1$  hyperedges, after removing the hyperedges in  $P$  from  $B$  in the bipartite graph  $G$  constructed earlier, the vertices in  $A$  has degree at least  $\lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - (n - 1)$ . The vertices in  $B$  has degree  $\binom{k}{2}$ , thus by Hall's marriage theorem, if  $\binom{k}{2} \leq \lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - (n - 1)$ , then there is a complete matching from  $A$  into  $B$ . By Lemma 6, the inequality is satisfied. Thus we can choose a hyperedge for each remaining edge of  $C'$  such that all the hyperedges chosen (including those from  $P$ ) are distinct. Therefore by using these hyperedges, we have an  $\ell$ -cycle  $C$  of  $T$  through  $x$ . As  $\ell$  and  $x$  are arbitrary,  $T$  is vertex-pancyclic.

Assume now  $k = 3$ . We repeat the first part of the proof. Again,  $P$  may have as many as  $n - 1$  hyperedges. However, for each pair of vertices, by Lemma 5, at most 4 of its corresponding hyperedges are in  $P$ . Therefore we require  $\binom{k}{2} \leq \lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - 4$  instead, which is satisfied by Lemma 6.  $\square$

*Remark.* Gutin and Yeo [2] proved that for  $k \geq 3$ , a hypertournament with  $n \geq k + 1$  has a Hamiltonian path and one with  $n \geq k + 2$  has a Hamiltonian cycle (Theorem 3). These bounds are tight: It is obvious that there are no Hamiltonian paths for  $n = k$ ; Gutin and Yeo constructed a strong  $k$ -tournament with  $n = k + 1$  vertices yet admits no Hamiltonian cycle.

It remains to be determined whether a strong  $k$ -tournament with  $n = k + 2$  vertices is vertex-pancyclic; or if there is a counter-example. Also, there are a few more cases for  $k < 8$  to be worked out.

To prove the following lemma, we introduce the calculus of finite differences: The finite forward difference  $\Delta_x f(x)$  of a function  $f(x)$  with respect to  $x$  is defined as  $\Delta_x f(x) = f(x + 1) - f(x)$ . The  $n$ -th finite difference is defined inductively as  $\Delta_x^n f(x) = \Delta_x(\Delta_x^{n-1} f(x))$ . If the function has more variables, they are held constant. This is the discrete analog of the derivative.

**Lemma 8.** *If  $n \geq k + 2d + 1$  for  $k \geq 8$ , then*

$$d \left[ \binom{k}{2} + (n - 1) \right] \leq \left\lceil \frac{1}{2} \binom{n-2}{k-2} \right\rceil.$$

*Proof.* Suppose  $k \geq 8$  and  $n \geq k + 2d + 1$ , we want to show that

$$f = f(n, k, d) = \binom{n-2}{k-2} + 1 - 2d \left[ \binom{k}{2} + n - 1 \right]$$

is nonnegative. Now

$$\begin{aligned} \Delta_n f(n, k, d) &= \binom{n-1}{k-2} + 1 - 2d \left[ \binom{k}{2} + n \right] \\ &\quad - \left( \binom{n-2}{k-2} + 1 - 2d \left[ \binom{k}{2} + n - 1 \right] \right) \\ &= \binom{n-2}{k-2} + \binom{n-2}{k-3} - 2dn - \left( \binom{n-2}{k-2} - 2dn + 2d \right) \\ &= \binom{n-2}{k-3} - 2d, \end{aligned}$$

which is nonnegative when  $n \geq k + 2d + 1$  and  $k \geq 8$ . Therefore it remains to show that  $f$  is nonnegative when  $n = k + 2d + 1$ .

To show that

$$g(k, d) = f(k + 2d + 1, k, d) = \binom{k+2d-1}{k-2} + 1 - 2d \left[ \binom{k}{2} + k + 2d \right]$$

is nonnegative, we shall show that  $g(k, 1)$ ,  $\Delta_d g(k, 1)$ ,  $\Delta_d^2 g(k, 1)$  and  $\Delta_d^3 g(k, d)$  are all nonnegative for  $k \geq 8$ .

Now  $g(k, 1) = \binom{k+1}{k-2} + 1 - 2 \left[ \binom{k}{2} + k + 3 \right]$  reduces to the case of  $n = k + 3$  considered in Lemma 6, and is nonnegative.

Now note that

$$\binom{k+2d+1}{k-2} = \binom{k+2d+1}{2d+3} = \frac{k+2d+1}{2d+3} \frac{k+2d}{2d+2} \binom{k+2d-1}{2d+1},$$

hence

$$\begin{aligned} \Delta_d g(k, d) &= g(k, d+1) - g(k, d) \\ &= \binom{k+2d+1}{k-2} - 2 \left[ \binom{k}{2} + k \right] - 4(d+1)^2 - \binom{k+2d-1}{2d+1} + 4d^2 \\ &= \left( \frac{k+2d+1}{2d+3} \frac{k+2d}{2d+2} - 1 \right) \binom{k+2d-1}{2d+1} - k^2 - k - 8d - 4, \\ \Delta_d g(k, 1) &= \left( \frac{k+3}{5} \frac{k+2}{4} - 1 \right) \binom{k+1}{3} - k^2 - k - 8 - 4 \\ &= \frac{1}{120} (k^5 + 5k^4 - 15k^3 - 125k^2 - 106k - 1440), \end{aligned}$$

which has only one real root between 5 and 6, and is 37 when  $k = 6$ . Similarly,

$$\begin{aligned} \Delta_d^2 g(k, d) &= \Delta_d g(k, d+1) - \Delta_d g(k, d) \\ &= \left[ \left( \frac{k+2d+3}{2d+5} \frac{k+2d+2}{2d+4} - 2 \right) \binom{k+2d+1}{2d+3} \frac{k+2d}{2d+2} + 1 \right] \binom{k+2d-1}{2d+1} - 8, \\ \Delta_d^2 g(k, 1) &= \left[ \left( \frac{k+5}{7} \frac{k+4}{6} - 2 \right) \binom{k+3}{5} \frac{k+2}{4} + 1 \right] \binom{k+1}{3} - 8 \\ &= \frac{1}{5040} (k^2 + 19k + 76)(k+1)k(k-1)(k-2)(k-3) - 8, \end{aligned}$$

which is clearly increasing when  $k \geq 4$ , and is 20 when  $k = 5$ . Finally,

$$\begin{aligned} \Delta_d^3 g(k, d) &= \Delta_d^2 g(k, d+1) - \Delta_d^2 g(k, d) \\ &= \frac{(k^2+8dk+17k+16d^2+56d+30)(k+4d+7)(k-2)(k-3)(k-4)}{(2d+7)(2d+6)(2d+5)(2d+4)(2d+3)(2d+2)} \binom{k+2d-1}{2d+1}, \end{aligned}$$

which is clearly nonnegative for  $k \geq 4$ ,  $d \geq 1$ . Therefore  $g(k, d) \geq 0$  for  $k \geq 8$ , and  $f(n, k, d)$  is increasing in  $n$  for  $n \geq k + 2d + 1$ , we have that  $f$  is indeed nonnegative for  $n \geq k + 2d + 1$ ,  $k \geq 8$ , as desired.  $\square$

**Theorem 9.** *Let  $T$  be a  $k$ -tournament on  $n$  vertices. If  $n \geq k + 2d + 1$  for  $k \geq 8$  and  $n \geq 14d + 1$  for  $k = 3$ , then  $T$  is  $d$ -disjoint-vertex-pancyclic if and only if  $T$  is  $d$ -edge-connected.*

*Proof.* We review the proof of Theorem 7. It is again obvious that a  $d$ -disjoint-vertex-pancyclic  $T$  is  $d$ -edge-connected. Now assume that  $T$  is  $d$ -edge-connected. We again consider  $k > 3$  first. Fix a vertex  $x$  of  $T$  and a length  $\ell \in \{3, \dots, n\}$ , we shall find  $d$  disjoint  $\ell$ -cycles of  $T$  through  $x$ . We again construct the majority digraph  $T'$ . Previously, because  $T$  is strong, there exists a path  $P$  in  $T$  connecting a vertex  $x_1$  from the terminal component  $T'_t$  to a vertex  $x_2$  in the initial component  $T'_1$ . Now, because  $T$  is  $d$ -edge-connected,  $T$  has  $d$  edge-disjoint paths  $P_1, P_2, \dots, P_d$  connecting  $x_1$  to  $x_2$ . We add the edges of  $P_i$  to  $T'$  to obtain a strong semi-complete digraph  $D_i$ . Moon's theorem gives us an  $\ell$ -cycle  $C'_i$  through  $x$ . Now we choose corresponding hyperedges to get  $d$  disjoint  $\ell$ -cycles of  $T$  through  $x$ . The edges arising only from the  $P_i$ 's can be directly used, as before. Since  $P_i$  has at most  $n - 1$  hyperedges,  $P_1, \dots, P_d$  can use at most  $d(n - 1)$  hyperedges. Since each remaining edge of  $T'$  corresponds to  $\lceil \frac{1}{2} \binom{n-2}{k-2} \rceil$  hyperedges, after removing the hyperedges in the  $P_i$ 's from  $B$  in the bipartite graph  $G$ , the vertices in  $A$  has degree at least  $\lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - d(n - 1)$ . Now we want to choose  $d$  corresponding hyperedges for each of the remaining edges of  $T'$ , such that all edges are disjoint. This can be accomplished by replicating  $A$   $d$  times. Let  $A' = \{(a, i) : a \in A, 1 \leq i \leq d\}$ , and join  $(a, i) \in A'$  with  $b \in B$  if and only if  $a$  and  $b$  are joined in  $G$ . Consider this new bipartite graph with partite classes  $A'$  and  $B$ . Now each vertex in  $B$  has degree  $d \binom{k}{2}$ . By Hall's marriage theorem, if

$$d \binom{k}{2} \leq \lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - d(n - 1),$$

which is satisfied by Lemma 8, then there is a complete matching from  $A'$  into  $B$ . Thus for each edge  $e$  of  $T'$ , we get  $d$  hyperedges of  $T$  by taking those matched to  $(e, i)$ ,  $1 \leq i \leq d$ . Now as these hyperedges are all distinct, and disjoint from those of  $P_1, \dots, P_d$ , we can use these to form edge-disjoint  $\ell$ -cycles  $C_1, \dots, C_d$  through  $x$  in  $T$ .

For  $k = 3$ , Lemma 5 allows us to replace  $n - 1$  with 4 in the above proof. Now  $d \binom{k}{2} \leq \lceil \frac{1}{2} \binom{n-2}{k-2} \rceil - 4d$  becomes  $3d \leq \lceil \frac{1}{2}(n - 2) \rceil - 4d$ , which is satisfied when  $n \geq 14d + 1$ .  $\square$

#### 4. REMARKS

Except for Lemma 5, all other results are proved in the past month. The progress so far are in line with the expectation that strong connectivity and vertex-pancyclicity are intimately related for hypertournaments. For  $k \geq 8$ ,  $d = 1$ , the only case left to consider is  $n = k + 2$ . However, we have not been able to prove it or find a counter-example. Another difficulty is solving inequalities involving binomial coefficients; often, there is no elegant way of solving it save applying the calculus of finite differences repeatedly. For the rest of the project, we shall continue to work on the same problem: The  $n = k + 2$  case mentioned above, and the generalization to  $d$  not necessarily 1. My mentor and I meet twice a week for about one hour each: I present new results and we discuss possible actions to take.

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