HAMILTONIAN CYCLES IN TOURNAMENTS

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1. Definitions

Let V be a n-set (set of size n). Let E be the collection of all possible k-subsets (subsets of size k) of V ($2 \leq k \leq n$) each taken in one of its $k!$ possible permutations. A pair $T = (V, E)$ is called a *hypertournament*, or a *k-tournament*. Each element of V is a *vertex*, and each ordered k-tuple of E is a *hyperedge* or, simply, an *edge*. For vertices $u, v \in V$ and an edge $e = (x_1, \ldots, x_k) \in E$, we say u *dominates* v via edge e if u precedes v in e, that is if $u = x_i$, $v = x_j$, $1 \leq i < j \leq k$. We denote this by uev. A *path* consists of an alternating sequence

$$x_0 e_1 x_1 e_2 x_2 \ldots x_{k-1} e_l x_l$$

of distinct vertices x_i and distinct edges e_i so that x_{i-1} dominates x_i via e_i ($i = 1, \ldots, l$). Such a path has *length l*. A *cycle* is a path when all vertices are distinct except $x_0 = x_l$. A path (cycle) of T is *Hamiltonian* if it contains all vertices of T. A k-tournament T is *pancyclic* if it contains cycles of all possible lengths. It is *vertex-pancyclic* if each vertex of T is contained in cycles of all possible lengths. A k-tournament T is *strong* if there is a path from u to v for each pair $u, v \in V$. The vertex set V and the edge collection E is also denoted $V(T)$ and $E(T)$, respectively. A hypertournament T is *d-edge-connected* if, for any two vertices $u,\ v \in V$, there are d pairwise edge-disjoint paths from x to y.

2. Known Results

Gutin and Yeo [1] proved the following.

Theorem 1.
(i) *Every k-tournament on n vertices* ($3 \leq k \leq n-1$) *has a Hamiltonian path.*
(ii) *Every strong k-tournament on n vertices* ($3 \leq k \leq n-2$) *has a Hamiltonian cycle.*

Recently, Petrovic and Thomassen [2] proved the following generalization of (ii).

Theorem 2. *Let T be a d-edge-connected k-tournament on n vertices. If $n \geq k+1+24d$ for $k \geq 4$, and $n \geq 30d+2$ for $k=3$, then T has d edge-disjoint Hamiltonian cycles.*

3. Unsolved Problems

Gutin and Yeo [1] mentioned as unsolved the problem of deciding if a k-tournament is pancyclic. Petrovic and Thomassen [2] characterized the pancyclic k-tournaments: If $k \geq 4$ and $n \geq k+25$ or if $k=3$ and $n \geq 32$, then T is vertex-pancyclic if and only if T is strong.

This characterization is incomplete in that it is only for n large compared to k. Furthermore, this is claimed without explaination. My first task in the beginning of the summer shall be to deduce this claim from the proof of Theorem 2. Then I shall attempt to solve the general problem for n not necessarily sufficiently large.

A related problem is whether vertex-connectivity 10^{10} implies two edge-disjoint Hamiltonian cycles; see [2, 3, 4]. For $k \geq 2$, a graph G is said to have vertex-connectivity k when it has at least $k+1$ vertices and the removal of $k-1$ vertices does not result in a disconnected graph. I can also explore other related problems concerning hypertournaments and edge-disjoint Hamiltonian cycles.

4. Exposure and Experiences

I am a sophomore majoring in mathematics. I have taken Ma 121a, and will have finished Ma 121b and c by the summer. I have also attended several seminars in combinatorics on Thursday mornings. I find combinatorics, especially graph theory, very interesting and would like to have further exposure and experience working on hypertournaments and graph theory in general this coming summer.

References

[1] G. Gutin and A. Yeo, Hamiltonian paths and cycles in hypertournaments, J Graph Theory 25 (1997), 277-286.
[2] V. Petrovic and C. Thomassen, Edge-disjoint Hamiltonian cycles in hypertournaments, J Graph Theory 51 (2006), 49-52.
[3] C. Thomassen, Problem 1, Discrete Math 36 (1981), 231.
[4] C. Thomassen, Edge-disjoint Hamiltonian paths and cycles in tournaments, Proc London Math Soc 45 (1982), 151-168.